The moment of inertia of a 1.5-Sun-mass neutron star
(given its density function)
AND
Its angular momentum
(if rotating at 700 revolutions per second)

Supplementary calculations for "Core collapse" in Chapter 6 of

The Nature of Gravitational Collapse

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The Book states as follows:

> …
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> Now, if the neutron star that emerges from the core collapse as described [in the Section "Core collapse" in Chapter 6] has a spin rate of 700 revolutions per second, how much rotational momentum is it able to accommodate? We know the mass to be 1.5 Solar masses; if it were much less than this, then it would not be, and could not be, a neutron star. We know the density distribution; according to the astrophysics experts, it ranges *linearly* from about 10^9 kilograms per cubic meter, at the surface, to 8×10^{17} kilograms per cubic meter, at the center. From this set of knowns, the spherical radius of the star can be calculated. It turns out to be 15.3 kilometers. (Keep in mind, this is only a rough approximation. Ignored is the fact that the neutron star's shape, because of the significant spin, is oblate and not spherical.) With a spin rate of 700 revolutions per second, this neutron structure would have an angular momentum of 8.24×10^{41} kilograms·meter^2 per second. That answers the question of how much rotational momentum a minimal neutron star is allowed to have just after its formation.
>
> In relative terms, this is a remarkably small amount, for it represents only a little more than five percent of the total momentum (1.5×10^{43} kg·m^2/s) that needs to be conserved or dissipated.

The calculation of Moment of Inertia of a 1.5⊙ neutron star (given its density function) and its angular momentum (if rotating at 700 revolutions per second) follows:

Here is how the angular momentum value —8.24×10^{41} kilograms·meter^2 per second— in the text was calculated:

First we need an expression for MASS in terms of the neutron star radius R (given only its density function, ranging *linearly* from ~10^9 kg/m^3, at the surface, to ~8×10^{17} kg/m^3, at the center [per Wikipedia: *neutron star*]).

Consider elemental spherical mass shells.

(Mass of thin shell) = (area of thin shell)·(thickness)·(density).

(Mass of shell @r) = (area of shell @r)·(dr)·(density @r).

Substitute the given linear density function (as defined by the linear density graph of Fig. S1-1), which we may approximate as $\left(-(8\times10^{17}/R)\times r + (8\times10^{17})\right)$ kg/m^3.

$$dM_{ns} = 4\pi(r)^2\,(dr)\,(-(8\times10^{17}/R)\times r + 8\times10^{17}), \text{ where } R \text{ is constant and } 0 \le r \le R,$$

$$= 4\pi(8\times10^{17})\,(-r^3/R + r^2)\,dr\,.$$

$$M_{\text{ns}} = 4\pi\left(8\times10^{17}\right)\int_{r=0}^{r=R}\left(-\frac{r^3}{R}+r^2\right)dr\ ,$$

$$= 4\pi\left(8\times10^{17}\right)\left|-\frac{r^4}{4R}+\frac{r^3}{3}\right|_0^R\ ,$$

$= 4\pi\ (8\times10^{17})\ (-R^4/4R + R^3/3)$,

Total neutron mass = ⅓ π (8×10^{17} kg/m^3) R^3, with R expressed in meters.

$$\text{Then,}\quad R = \left(\frac{M_{\text{NeutronStar}}}{\tfrac{1}{3}\pi\left(8\times10^{17}\right)kg/m^3}\right)^{1/3},$$

which, when solved, equals: **15.3 km**.

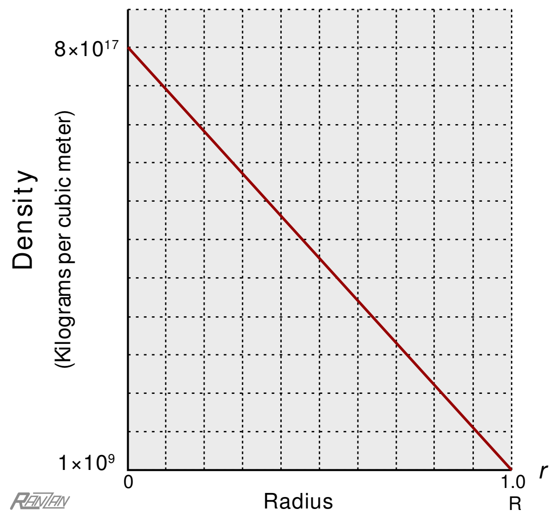

Fig. S1-1. Linear mass-density profile for the conventional neutron star.

Next, we find the moment of Inertia:

The moment of Inertia of an elemental spherical mass shell, about any diameter, is known to be:

(Rotational Inertia of thin shell) = (2/3)·(mass of thin shell)·(radius)^2,

Restated for an elemental shell within the neutron star whose linear density profile is given as having a range from ~1.0×10^9 kg/m^3, at the surface, to ~8.0×10^{17} kg/m^3, at the center (Fig. S1-1).

(Inertia$_{\text{thin shell}}$ @r) = (2/3) (Area$_{\text{shell}}$ @r)·(Thickness$_{\text{shell}}$)·(Density @r)·(Radius$_{\text{shell}}$)^2, where r is the radius.

Make the following substitutions:

Area of the shell = $4\pi\ r^2$;
Thickness of the shell = dr;
Density function = $((-8\times10^{17}/R)\times r\ + (1\times10^9))$ kg/m^3 ≈ $((-8\times10^{17}/R)\times r\ + (8\times10^{17}))$ kg/m^3.

The element of rotational inertia, then, looks like this (except for the missing density units, kg/m^3, which will be added later):

$dI_{\text{ns}} = (2/3)\ (4\pi\ r^2)\ (dr)\ \big((-8\times10^{17}/R)\times r\ + (8\times10^{17})\big)\ (r^2)$, where R is constant and $0 \le r \le R$,

which simplifies to

$= (2/3)\ (4\pi\ r^2)\ (8\times10^{17})\ \big((-r/R) + 1\big)\ (r^2)\ (dr)$,
$=\ (8/3)\ \pi\ (8\times10^{17})\ (-r^5/R + r^4)\ dr$.

Then, by integrating between the limits of r:

$$I_{\text{ns}} = \frac{8}{3}\pi\left(8\times10^{17}\right)\int_{r=0}^{r=R}\left(-\frac{r^5}{R}+r^4\right)dr\ ,$$

$$= \frac{8}{3}\pi\left(8\times10^{17}\right)\left|-\frac{r^6}{6R}+\frac{r^5}{5}\right|_0^R\ ,$$

$$= \frac{4}{45}\pi\left(8\times10^{17}\right)R^5\left(kg/m^3\right).$$

When evaluated for the 1.5☉ neutron star (radius R of 15.3×10^3 meters), this gives a **moment of inertia equal to 1.873×10^{38} kg·m^2**.

Lastly, the rotational momentum:

Rotational momentum = (moment of inertia) × (angular velocity),

$$L_{ns} = \left(\frac{4}{45} \pi \left(8 \times 10^{17} \right) R^5 \left(kg / m^3 \right) \right) \times \left(700 \times 2\pi / s \right) .$$

If this same star is rotating at 700 times per second, and if treated as a rigid body, it will have an angular momentum of **8.24×10^{41} kg·m²/s**.

As pointed out in Chapter 6, this is 5.5 % of the original total momentum (1.5×10^{43} kg·m^2/s) that needs to be conserved or dissipated.

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